Appendix 5.2
Mathematical Expressions
for Pile Friction

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The mathematical expressions for representation of pile friction were based on the Cross Border Link Study and the Delft 3D-FLOW module developed by Delft Hydraulics. A quadratic friction term added to the momentum equations can be expressed in the form:

Friction loss in the u coordinate direction = $[C_{loss,u}\, U\, |\langle U \rangle|\,] / \Delta x$ (m/s^2)
Friction loss in the v coordinate direction = $[C_{loss,v}\, V\, |\langle U \rangle|\,] / \Delta y$ (m/s^2) (5.1)

Where:

$C_{loss,u}$ and $C_{loss,v}$	= loss coefficients in the u and v coordinate directions;
$\langle U \rangle$	= velocity vector (U, V);
$\lvert\langle U \rangle\rvert$	= magnitude of the velocity vector = $\sqrt{U^2+V^2}$ (m/s); and
Δx and Δy	= grid distances in u and v coordinate directions respectively (m).

According to the water speeds in each model layer, the additional quadratic friction term influences the horizontal flow distribution in each layer and so indirectly affects the vertical turbulent exchange.

The force exerted on the vertical section (Δz) of one pile can be expressed as:

Drag force on a pile in the u coordinate direction:

$$F_u = C_d \frac{1}{2} \rho\, D\, U_e |\langle U_e \rangle| \Delta z$$

Drag force on a pile in the v coordinate direction:

$$F_v = n\, C_d \frac{1}{2} \rho\, D\, V_e |\langle U_e \rangle| \Delta z \qquad (5.2)$$

Where:

C_d	= drag coefficient;
ρ	= density of water (kg/m^3);
$\langle U_e \rangle$	= effective approach velocity vector (U_e, V_e) (m/s);
$\lvert\langle U_e \rangle\rvert$	= magnitude of the effective approach velocity vector = $\sqrt{U^2+V^2}$ (m/s);
D	= diameter of the pile (m); and
Δz	= length of the vertical section (m).

The effective approach velocity can be calculated using the wet cross section as seen in flow direction and is expressed as:

Effective approach velocity $\langle U_e \rangle = \langle U \rangle \times [A_T / A_e] = \langle U \rangle \times a$ (5.3)

Where:

A_T = total cross-sectional area (m^2);
A_e = effective wet cross sectional area
= total cross-sectional area (A_T) - area blocked by the piles (m^2); and
a = ratio of the total area to the effective area.

Assuming the piles are not in the shadow of each other, the total force exerted on the vertical section for n numbers of piles can be expressed as:

Total drag force in the u coordinate direction:

$$F_{tot,u,} = n\, C_d \frac{1}{2} \rho\, D\, U_e |\langle U_e \rangle| \Delta z$$

Total drag force in the v coordinate direction:

$F_{tot,v} = n\, C_d \frac{1}{2} \rho\, D\, V_e \,|< U_e >|\, \Delta z$

Where: n = number of piles in the control grid cell

The total friction loss term in the u and v coordinate directions can be determined by dividing the forces by the mass in the control volume (= $\rho\, \Delta x \Delta y \Delta z$) and can be expressed as:

Total friction loss in the x-direction = $n\, C_d \frac{1}{2} D\, U_e \,|< U_e >| / (\Delta x \Delta y)$

Total friction loss in the y-direction = $n\, C_d \frac{1}{2} D\, V_e \,|< U_e >| / (\Delta x \Delta y)$ (5.4)

Combining Equation (5.1) and Equation (5.4), the loss coefficients for n numbers of piles in the u and v coordinate directions are:

Loss coefficient in the u coordinate direction $C_{loss,\,u} = [n\, C_d \frac{1}{2} D\, a^2\,] / (\Delta y)$

Loss coefficient in the v coordinate direction $C_{loss,\,v} = [n\, C_d \frac{1}{2} D\, a^2\,] / (\Delta x)$ (5.5)

Based on the equations (5.1) – (5.5), the loss coefficients were calculated for relevant model grid cells in both u and v directions for model input.